Published by Institute of Physics Publishing for SISSA/ISAS

Received: *May 3, 2004*
Revised: *May 19, 2004*
Accepted: *May 20, 2004*

Flat space physics from holography

Raphael Bousso

Center for Theoretical Physics, Department of Physics
University of California, Berkeley, CA 94720-7300, U.S.A., and
Lawrence Berkeley National Laboratory
Berkeley, CA 94720-8162, U.S.A.
E-mail: bousso@lbl.gov

Abstract: We point out that aspects of quantum mechanics can be derived from the holographic principle, using only a perturbative limit of classical general relativity. In flat space, the covariant entropy bound reduces to the Bekenstein bound. The latter does not contain Newton's constant and cannot operate via gravitational backreaction. Instead, it is protected by — and in this sense, predicts — the Heisenberg uncertainty principle.

Keywords: M-Theory, M(atrix) Theories, Models of Quantum Gravity, Black Holes.

Contents

1. Introduction

According to the holographic principle [1]–[4], classical spacetime geometry and its matter content arise from an underlying quantum gravity theory in such a way that the covariant entropy bound [5] — thus far only a well-supported conjecture relating matter entropy to the area of surfaces — is automatically satisfied. To some degree, this expectation was borne out by the AdS/CFT correspondence [6], which provides a non-perturbative, manifestly holographic definition of string theory in certain spacetimes [7].

By deriving consequences and implications of the holographic principle, we may learn how specific laws of physics will arise from quantum gravity even before we know the underlying theory in detail. An example of such an implication is the generalized second law of thermodynamics (GSL) [8, 9, 10]: the generalized covariant bound implies [11] that the total entropy of ordinary matter systems and black holes will never decrease in any physical process (assuming the ordinary second law holds), a conjecture widely believed to be true but difficult to prove by other means.

The purpose of this paper is to expose another such implication. Thus, we adopt the holographic relation between information and geometry as our axiomatic starting point:

$$S \leq \frac{\Delta A}{4 l_{\mathrm{Pl}}^2} \,. \tag{1.1}$$

This is a generalized version [11] of the covariant entropy bound. The area difference ΔA will be defined more carefully in section 2. For now we note only that the relation involves a fundamental unit of length, the Planck length, reflecting its origin in quantum gravity. (We set $k_{\mathrm{B}} = c = 1$.)

The entropy S is generally a measure of information; in quantum field theory, it is identified with the logarithm of the number of independent Fock space states compatible with the geometric boundary conditions.[1] But note that $\hbar$ (the value of the equal-time commutator of a field and its conjugate momentum) is not explicitly fixed by our assumption. It will be *derived*, using the Raychaudhuri equation of classical general relativity, which contains only Newton's constant G.

As we review in section 3, the Raychaudhuri equation allows us to eliminate Newton's constant from eq. (1.1) in regions where gravity is weak. This yields an important intermediate result, the Bekenstein bound [12], which involves only the combination l_{Pl}^2/G of physical constants.

In section 4 we show that the Bekenstein bound would be violated if the position and momentum uncertainties of a particle were allowed to become sufficiently small, i.e., if

$$\delta x \, \delta p \ll \frac{l_{\mathrm{Pl}}^2}{G} \,. \tag{1.2}$$

From this we conclude that physical states in Minkowski space must obey

$$\delta x \, \delta p \gtrsim \frac{l_{\mathrm{Pl}}^2}{G} \,. \tag{1.3}$$

This inequality is the the Heisenberg uncertainty relation. Thus, Planck's constant emerges as a derived quantity, expressed in terms of the geometric unit of information, l_{P}^2, and Newton's constant G:

$$\hbar \approx \frac{l_{\mathrm{Pl}}^2}{G} \,. \tag{1.4}$$

Of course, the relation eq. (1.4) is necessary for the first law of thermodynamics to be satisfied by the entropy $S = \pi R^2/l_{\mathrm{Pl}}^2$, temperature $T = \hbar/4\pi R$, and mass $M = R/2G$ of a Schwarzschild black hole. Indeed, the calculation of the Hawking temperature remains the only known semiclassical method for calibrating the numerical coefficients. But that computation takes quantum field theory as a starting point. Here we argue that the Planck constant can be obtained directly starting from a principle of quantum gravity.

How can it be legitimate to use classical general relativity to derive a key aspect of quantum mechanics? Einstein's theory offers only an approximate description of Nature, which will surely be transcended when the problem of unifying quantum theory and gravity is solved. Quantum mechanics, however, is often assumed to be immune to this fate. This prejudice — implicit as soon as we try to "quantize gravity" — is not without merit: It has proven difficult to modify quantum mechanics sensibly. More importantly, string theory is a perfectly quantum mechanical theory which does include gravity. However, it is not clear how much of string theory has been explored, and how it may be related to a realistic universe.

[1] We stress this distinction since we do not wish to assume that the unified theory underlying eq. (1.1) is in fact quantum mechanical in nature, but rather to leave open the possibility that it may encode information by means other than the states of a Hilbert space.

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Indeed, there are reasons to question the assumption that quantum mechanics is universal. It is unclear how to ascribe operational meaning to quantum mechanical amplitudes in highly dynamical spacetime regions, because experiments cannot be repeated. Such difficulties become exacerbated at spacelike singularities. For quantum mechanical evolution to proceed, a time coordinate would have to be singled out among the directions of spacetime. It would have to survive the crunch and live on as a quantum mechanical evolution parameter. But near a generic singularity, time breaks down as a geometric object no less than space does. Attempts to resolve non-timelike singularities in string theory [13]–[16] have so far only yielded evidence that this covariant behavior persists.

In the absence of a sufficiently general definition of quantum mechanical observables, it therefore remains conceivable — and perhaps plausible — that quantum mechanics is no more fundamental than classical spacetime, and that they both emerge from a unified description only in certain limits. The arguments presented in the present paper are consistent with this viewpoint. Note that our derivation applies only in weakly gravitating regions, leaving open the possibility that in some backgrounds (e.g., in cosmology or gravitational collapse) quantum mechanics may not emerge in its conventional form.

2. Entropy bounds

In this section we introduce various entropy bounds and the holographic principle. We explain why we choose eq. (1.1) as our axiomatic starting point. This is a review section and is not itself part of the derivation. Thus, we will permit ourselves to write the Bekenstein bound in terms of $\hbar$, and to make occasional use of the relation $l_{\mathrm{Pl}}^2 = G\hbar$ for the purpose of elucidating the properties of the bounds.

2.1 Covariant bound

The covariant entropy conjecture [5] applies to any spacetime region that is well described by classical general relativity. It states that the entropy of matter on any light-sheet L of any two-dimensional surface B obeys

$$S(L) \leq \frac{A}{4l_{\mathrm{Pl}}^2}\,, \tag{2.1}$$

where A is the area of B. A light-sheet is a 2+1-dimensional hypersurface generated by nonexpanding light rays orthogonal to B. A full review is found in ref. [4].

Any surface B has four orthogonal null directions, at least two of which are nonexpanding and give rise to light-sheets. For example, a spherical surface in Minkowski space has two light-sheets corresponding to past and future light-cones ending on B (figure 1). When neighboring light-rays intersect, the expansion becomes positive, and the generating light-rays must be terminated. This is why the light-sheets in figure 1 stop at the tips of the cones.

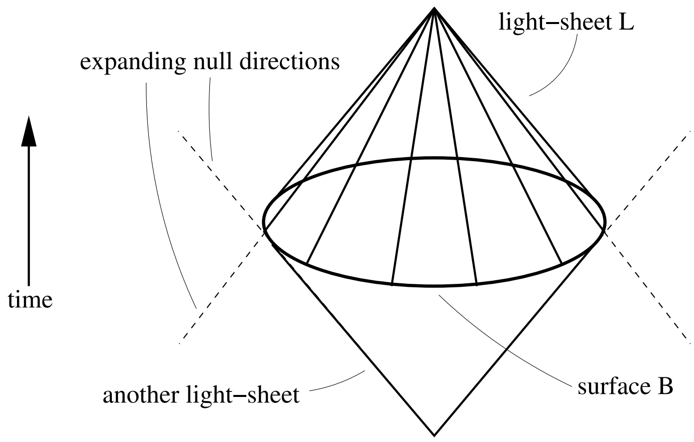

Figure 1: A two-dimensional surface and its light-sheets.

Gravitational backreaction plays a crucial role in preventing violations of this bound. In realistic systems, an increase in entropy is accompanied by an increase in energy. Energy focusses light rays by an amount proportional to G. Thus it hastens the termination of a light-sheet at caustic points, preventing it from "seeing" too much entropy.

We have no quantitative explanation why this backreaction should always suffice for the holographic bound. Ultimately this will have to be explained by an underlying theory in which information, rather than matter, is fundamental. But at a qualitative level, the notion of gravitational focussing appears to capture a key aspect of the bound.

Decreasing Newton's constant suppresses the gravitational backreaction and allows more mass and entropy on the light-sheet. But the bound, $A/4G\hbar$, is inversely proportional to G and hence compensates by becoming more lenient as $G \to 0$, and trivial for $G = 0$.

2.2 Bekenstein bound

A version of Bekenstein's "universal entropy bound" [12] appears as a key intermediate result in our argument. In its original form, the bound states that the entropy S of any weakly gravitating matter system obeys

$$S \leq \frac{2\pi M R}{\hbar}, \tag{2.2}$$

where M is the total gravitating mass of the matter and R is the radius of the smallest sphere that barely fits around the system.

The absence of Newton's constant in Bekenstein's bound is notable. It renders eq. (2.2) independent of the strength of gravity, G, in its regime of validity. In particular, the bound remains nontrivial when gravity is turned off completely ($G = 0$). Therefore, gravitational physics cannot play a role in upholding it, quite unlike the case of the holographic bound.

If one substitutes for M using the weak gravity condition

$$M \ll \frac{R}{G}, \tag{2.3}$$

it is immediately apparent that the Bekenstein bound, in its regime of validity, is significantly tighter than the covariant bound, eq. (2.1). To appreciate the difference, consider a single massive elementary particle. If we take M to be its rest mass, the particle can be localized to within a Compton wavelength: $R \approx \hbar/M$. Its entropy is of order unity: $S \approx 1$. The Bekenstein bound, $2\pi MR/\hbar$, is also of order one, and therefore is roughly saturated! The holographic bound, however, is given by the surface area in Planck units, $\pi R^2/l_{\rm Pl}^2$, which is typically a huge number. E.g., for an electron, the holographic bound gives $S \leq 10^{44}$, which is correct but rather uninteresting.

In recent years the holographic bound, despite its relative weakness, received far more attention than the Bekenstein bound. 't Hooft [1] and Susskind [2] ascribed fundamental significance to a bound in terms of area, asserting that the number of degrees of freedom in quantum gravity is given by the area of surfaces in Planck units — a conjecture that received strong support by subsequent developments in string theory [6]. Moreover, their holographic bound has turned out to admit the covariant formulation (2.1), whose apparent validity in strongly gravitating regions provides the most convincing evidence yet for the significance of the holographic principle in all spacetimes.

The Bekenstein bound has lacked a similar interpretation as a direct imprint of fundamental physics. It has been regarded mainly as a practical limit on information storage and transfer. Moreover, it contains quantities (energy, radius) which are well-defined only in special backgrounds. This appeared to preclude its generalization to arbitrary spacetimes — a prerequisite for a fundamental role. The result of ref. [17], however, clarifies that the Bekenstein bound should properly be viewed as expressing the constraints placed by the holographic principle on the physics of flat space.

2.3 Generalized covariant bound

The Bekenstein bound is much tighter than the covariant bound, but also much less generally applicable. Thus it is clear that neither bound can imply the other. This has obscured the relationship of Bekenstein's bound to the holographic principle.

Soon after it was first proposed, it was noticed [11] that the covariant bound implies the generalized second law of thermodynamics (GSL) [8, 9] for matter that collapses to form a black hole. However, in the form eq. (2.1) the covariant bound is not strong enough to imply the GSL for matter that is added to an existing black hole. This motivated Flanagan, Marolf, and Wald [11] to write down a stronger version of the covariant bound,[2] from which the GSL does follow in all cases.

Consider a partial light-sheet ΔL, which is terminated prematurely, before the light-rays self-intersect (figure 2). In general, ΔL will not capture as much matter (and as much entropy) as the fully extended light-sheet. On a partial light-sheet, the final area spanned

[2] In ref. [11] the emphasis was on showing that certain local conditions on entropy density and energy density are sufficient for the (generalized) covariant bound (see also ref. [18]). Here we make no reference to such phenomenologically motivated assumptions and derivations. Rather, we will consider the GCEB as an axiomatic starting point.

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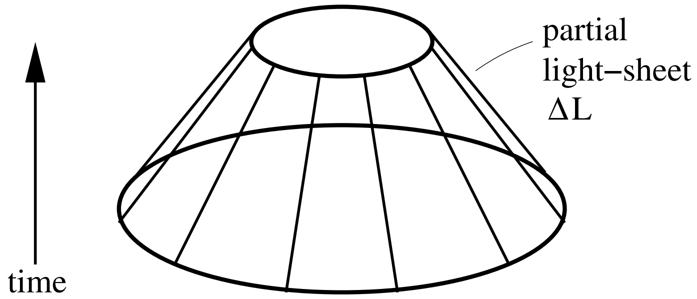

Figure 2: Partial light-sheet contracting from area A to A'.

by the light-rays, A', will be nonzero and, by the nonexpansion condition, $A' \leq A$. Thus it is natural to conjecture a "generalized" covariant entropy bound (GCEB),

$$S(\Delta L) \leq \frac{A - A'}{4l_{\mathrm{Pl}}^2} \,. \tag{2.4}$$

When this inequality is applied to weakly gravitating systems, it does in fact imply the Bekenstein bound [17], as we will now show.

3. Derivation of the Bekenstein bound

In this section, we show that the GCEB implies the Bekenstein bound. Instead of reproducing the rigorous derivation recently given in ref. [17], we will present a simplified argument which captures the main idea and yields the Bekenstein bound up to factors of order unity.

Consider an arbitrary, weakly gravitating system of mass M, which fits into a sphere of radius R — for example, the earth. In order to apply the covariant bound, one has to construct a light-sheet that intersects the worldvolume of the earth. There are many possibilities, and most will not lead to interesting results. For example, one could begin a light-sheet on the surface of the earth and follow the null geodesics towards the center of the earth. But then the final area on the light-sheet would vanish: $A' = 0$. Thus, we would fail to exploit the power of the tighter bound (2.4). We would learn merely that the entropy is smaller than the surface area of the earth, which is a far weaker statement than the Bekenstein bound.

A better strategy is to take advantage of the fact that the covariant bound does not require the initial surface A to be closed. This allows us to "X-ray" the system from the side. Consider parallel light rays which are emitted from a flat disk[3] tangential to the earth, with area $A = \pi R^2$. An image plate of equal area A may be placed on the opposite side of the earth (figure 3).

[3]More precisely, one must choose a surface on which the initial expansion of the light rays vanishes exactly. For weakly gravitating systems, this can always be arranged by deforming the disk slightly to compensate for deviations from flat space [17].

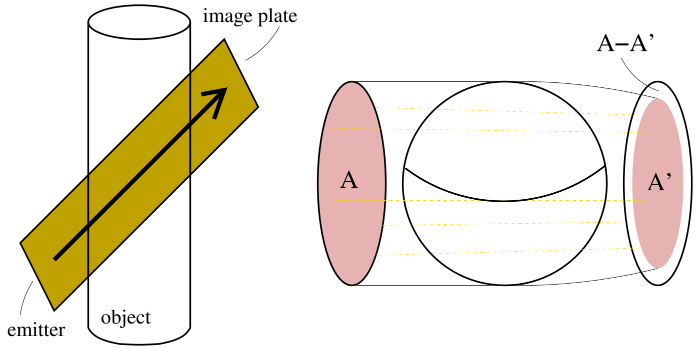

Figure 3: X-raying a weakly gravitating object. Left: spacetime view; right: spatial view.

We have arranged for the "X-rays" to start out exactly parallel to each other. But as they traverse the earth, they will be focussed by its mass and hence will begin to contract. Hence, the final area, A', will be slightly smaller than A. The rays will not illuminate all of the final plate, but will miss an annulus of area $A - A'$. Because gravity is weak, this area is determined entirely by the contraction of the outermost light rays (the ones that just skim the earth's surface). It can be quickly estimated from the standard bending of light effect.

The deflection angle is of order GM/R. Hence, the width of the annulus is of order GM, and the area is

$$A - A' \approx GMR\,. \tag{3.1}$$

Thus, eq. (2.4) implies that the entropy of the system obeys

$$S \lesssim \frac{MR}{l_{\rm Pl}^2/G}\,. \tag{3.2}$$

This is the Bekenstein bound (up to factors of order one).[4] The full derivation [17] is independent of any assumptions about the shape of the object and the distribution of stress-energy. The agreement with the Bekenstein bound then includes the numerical prefactor.[5]

It should be stressed that our understanding of the Bekenstein bound is still imperfect. The bound is quite sensitive to the precise definition of the entropy contained in a finite region. As entropy is fundamentally a nonlocal concept, such definitions are fraught with difficulties. The derivation we have presented does not automatically resolve this problem, because the entropy S just goes along for the ride.

It will be especially important to understand whether the bound applies to any field theory satisfying reasonable energy conditions, or whether it imposes further restrictions on

[4]With the relation $l_{\rm Pl}^2 = G\hbar$, we could transform this result to the standard form (2.2). This would be appropriate if we were interested in testing the GCEB through tests of the Bekenstein bound, as in ref. [19]. Here, however, we take the GCEB to be true axiomatically and aim to derive eq. (1.4) from it. Hence we shall retain the form (3.2).

[5]For non-spherical systems, one finds that the result obtained from the GCEB is actually stronger than (2.2).

the lagrangian. For example, theories with an astronomical number of species appear to be incompatible with the bound in the formulations proposed in refs. [19, 20]. By extension, these questions apply also to the GCEB. They are less crucial for the original covariant bound. The argument in this paper is not sensitive to these issues, but it does assume that some rigorous formulation of the bounds exists.

4. Derivation of the uncertainty principle

Recall that so far, we have worked entirely in terms of a fundamental length scale, $l_{\rm Pl}$, which characterizes the maximal information content of the light-sheets of a given surface as a function of its area; and Newton's constant, G, which measures the geometric focussing power of the stress-energy of matter. We have not made any explicit use of quantum field theory.

To obtain finite microcanonical entropy, as demanded by the entropy bound, a discretization scheme for physical fields is necessary. The canonical equal-time quantization of a relativistic field introduces $\hbar$ as a constant that defines the commutator between a field operator and its canonical conjugate. For example, a scalar field obeys:

$$[\phi(\mathbf{x}), \dot{\phi}(\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}')\hbar\,. \tag{4.1}$$

In the one-particle sector of Fock space, this implies a nontrivial commutation relation between the position and momentum operators,

$$[\hat{x}, \hat{p}_x] = i\hbar\,, \qquad \text{etc.}, \tag{4.2}$$

which in turn leads to the Heisenberg uncertainty principle

$$\delta p_x \delta x \geq \frac{\hbar}{2}\,, \qquad \text{etc.}, \tag{4.3}$$

where $\delta x \equiv \langle(\hat{x}-\langle\hat{x}\rangle)^2\rangle^{1/2}$ and $\delta p_x \equiv \langle(\hat{p}_x - \langle\hat{p}_x\rangle)^2\rangle^{1/2}$. This inequality can be saturated by gaussian wavepackets.

A particle of rest mass m which is at least marginally relativistic ($\mathbf{p}^2 \gtrsim m^2$) has a dispersion relation

$$E = \sqrt{m^2+\mathbf{p}^2} \approx |\mathbf{p}|\,. \tag{4.4}$$

Consider a gaussian wavepacket with vanishing momentum expectation value, and with momentum uncertainty $\delta p_x \approx \delta p_y \approx \delta p_z$. Its mass is given by

$$M = \langle E\rangle \approx |\delta p_x|\,. \tag{4.5}$$

Hence, the particle obeys

$$M\delta x \approx \hbar\,. \tag{4.6}$$

The spatial size of the particle is approximately its position uncertainty, $R \approx \delta x$. The entropy of a typical particle is given by the logarithm of the number of states of different spin and hence is of order one. Even when several different particle species are allowed, one still obtains $S \approx 1$ with the fields we observe in Nature. Hence, eq. (4.6) would violate the bound (3.2) unless

$$\hbar \gtrsim \frac{l_{\rm Pl}^2}{G}\,. \tag{4.7}$$

The right hand side of the Heisenberg uncertainty relation (and of the canonical commutation relations) is thus determined from the holographic entropy-area relation and classical gravity.[6]

To turn the inequality (4.7) into an approximate equality, we invoke economy and assume that $\hbar$ should be chosen such that it becomes possible to saturate the Bekenstein bound approximately. The example of an elementary particle discussed in section 2.2 demonstrates that the bound can indeed be roughly saturated when $G\hbar/l_{\rm Pl}^2$ is set to unity.

To go further and obtain an exact equality, $\hbar = l_{\rm Pl}^2/G$, one would need at least one example of a system that precisely saturates the Bekenstein bound with this value. This is an important outstanding problem, along with the question of the proper definition of entropy in a strict formulation of the bound.

Acknowledgments

I would like to thank M. Aganagic, T. Banks, J. Bekenstein, E. Flanagan, D. Marolf, J. Polchinski and L. Susskind for discussions.

References

[1] G. 't Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026.

[2] L. Susskind, *The world as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089].

[3] R. Bousso, *Holography in general space-times*, *J. High Energy Phys.* **06** (1999) 028 [hep-th/9906022].

[4] R. Bousso, *The holographic principle*, *Rev. Mod. Phys.* **74** (2002) 825 [hep-th/0203101].

[5] R. Bousso, *A covariant entropy conjecture*, *J. High Energy Phys.* **07** (1999) 004 [hep-th/9905177].

[6] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[7] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, hep-th/9805114.

[8] J.D. Bekenstein: *Black holes and the second law*, *Nuovo Cim.* **4** (1972) 737.

[9] J.D. Bekenstein, *Black holes and entropy*, *Phys. Rev.* **D 7** (1973) 2333.

[10] J.D. Bekenstein, *Generalized second law of thermodynamics in black hole physics*, *Phys. Rev.* **D 9** (1974) 3292.

[6] Note that this is also implied by the more rigorous formulation of the Bekenstein bound recently proposed in ref. [20]. There the bound takes the form $S \leq 2\pi^2 K G\hbar/l_{\rm Pl}^2$, where K labels the Fock space sector of DLCQ. One immediately obtains $\hbar \gtrsim l_{\rm Pl}^2/G$ by specializing to $K=1$ and taking the logarithm of the number of particle species to be of order one ($S_{K=1} \approx 1$).

[11] E.E. Flanagan, D. Marolf and R.M. Wald, *Proof of classical versions of the bousso entropy bound and of the generalized second law*, *Phys. Rev.* **D 62** (2000) 084035 [hep-th/9908070].

[12] J.D. Bekenstein, *A universal upper bound on the entropy to energy ratio for bounded systems*, *Phys. Rev.* **D 23** (1981) 287.

[13] H. Liu, G. Moore and N. Seiberg, *Strings in a time-dependent orbifold*, *J. High Energy Phys.* **06** (2002) 045 [hep-th/0204168].

[14] H. Liu, G. Moore and N. Seiberg, *Strings in time-dependent orbifolds*, *J. High Energy Phys.* **10** (2002) 031 [hep-th/0206182].

[15] G.T. Horowitz and J. Polchinski, *Instability of spacelike and null orbifold singularities*, *Phys. Rev.* **D 66** (2002) 103512 [hep-th/0206228].

[16] A. Lawrence, *On the instability of 3d null singularities*, *J. High Energy Phys.* **11** (2002) 019 [hep-th/0205288].

[17] R. Bousso, *Light-sheets and Bekenstein's bound*, *Phys. Rev. Lett.* **90** (2003) 121302 [hep-th/0210295].

[18] R. Bousso, E.E. Flanagan and D. Marolf, *Simple sufficient conditions for the generalized covariant entropy bound*, *Phys. Rev.* **D 68** (2003) 064001 [hep-th/0305149].

[19] R. Bousso, *Bound states and the Bekenstein bound*, *J. High Energy Phys.* **02** (2004) 025 [hep-th/0310148].

[20] R. Bousso, *Harmonic resolution as a holographic quantum number*, *J. High Energy Phys.* **03** (2004) 054 [hep-th/0310223].